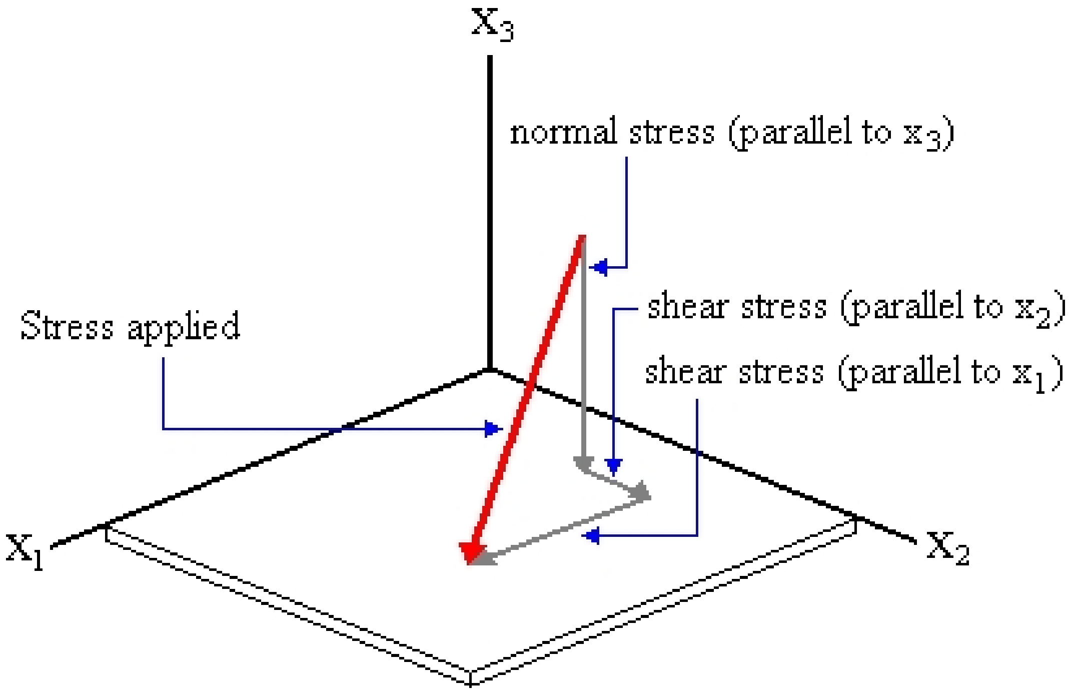
Transformation of Stress and Strain

Stress

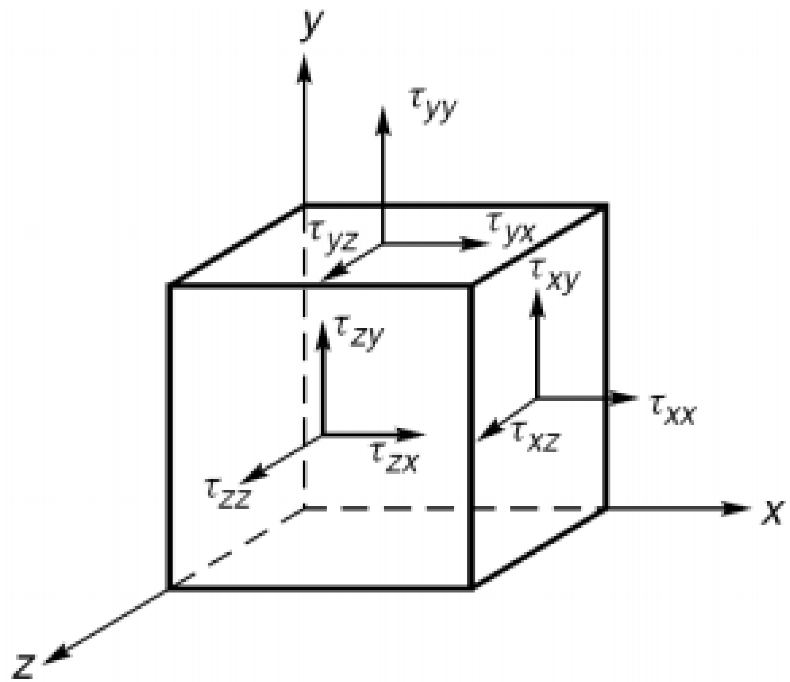
- Stress is a measure of the internal forces in a body between its constituent particles, as they resist separation, compression or sliding in response to externally applied forces.
- The mathematical definition of stress is defined as; the force exerted per unit area.

$$\vec{\sigma} = \frac{\vec{F}}{A}$$

Stress on a plane



Stress at a point



Stress Tensor

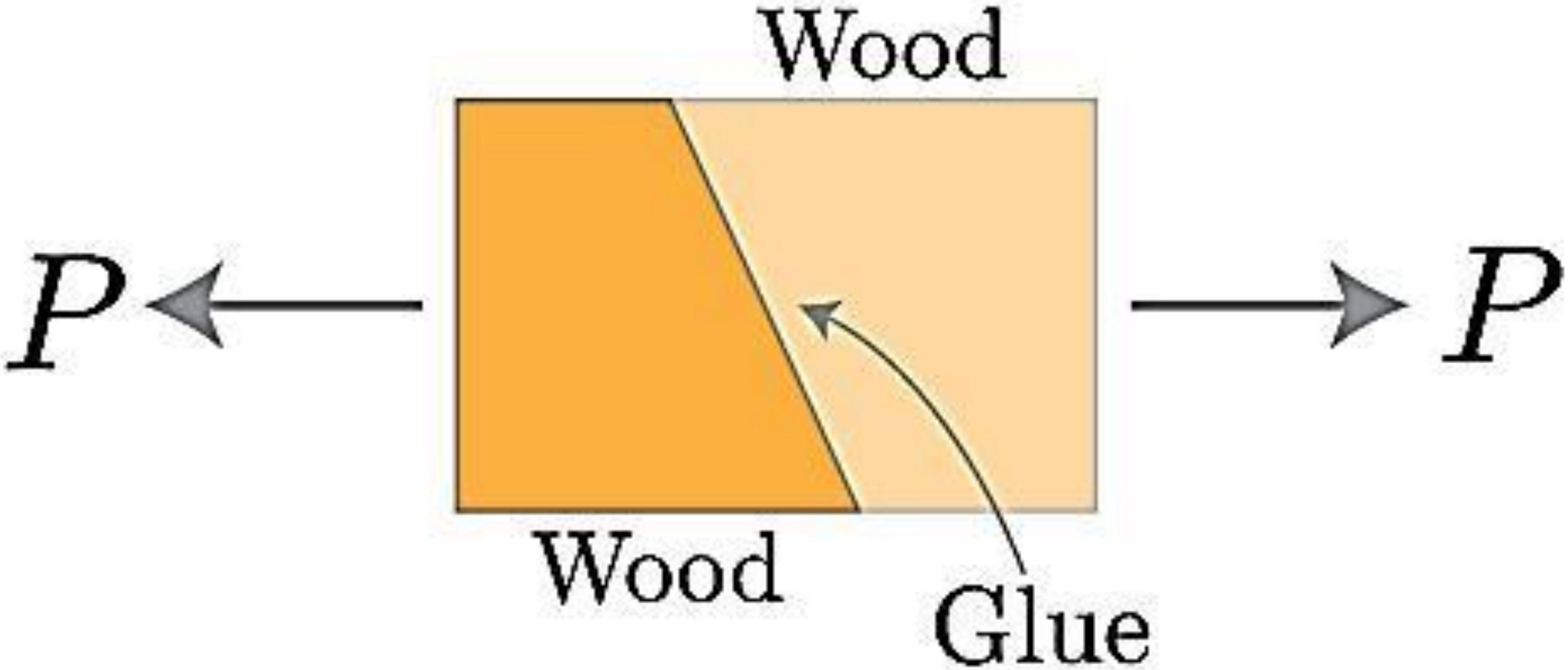
$$\sigma = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

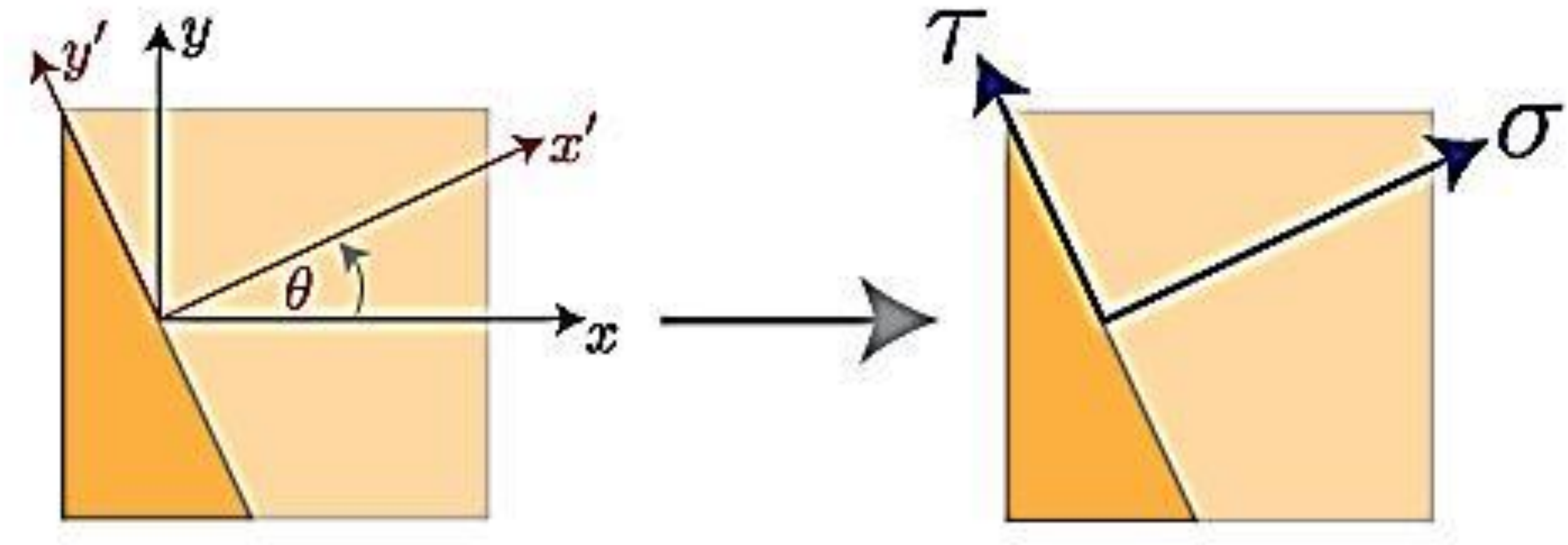
$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} = \left[\sigma_{ij}\right]$$

Strain Tensor

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_{zz} \end{bmatrix} = \left[\varepsilon_{ij} \right]$$

Transformation of stress and strain

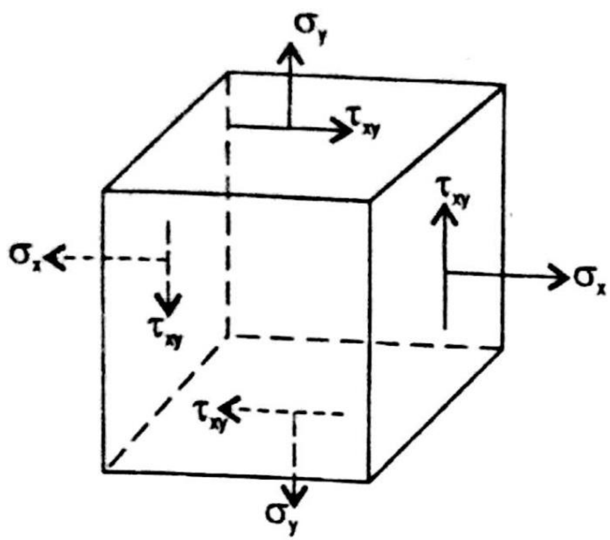

Wood
P
P
Wood
Glue



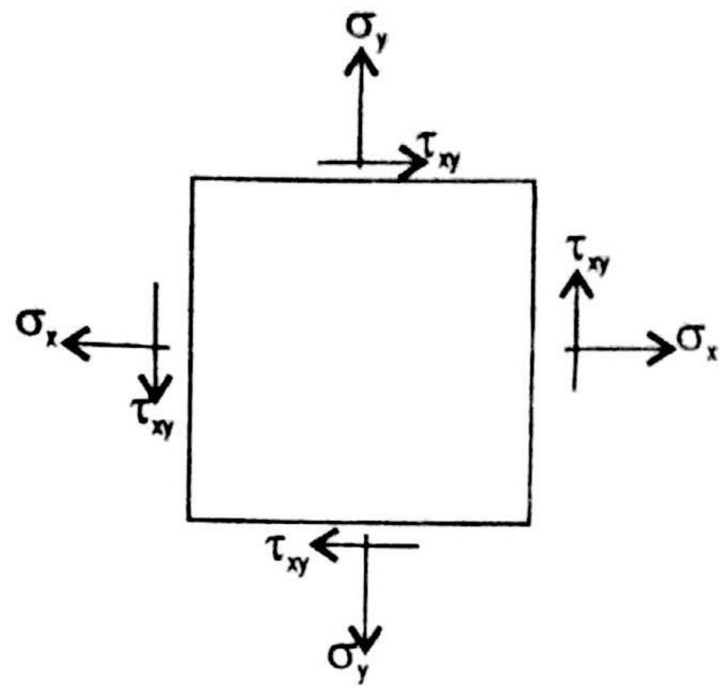
y'
y
x'
θ
x
τ
σ

Plane State of Stress

- Plane stress - In a given state of stress if σ_z=0, τ_{xz}=0 and τ_{yz}=0.



(a) Plane stress



(b) Conventional representation

- **STRESS ANALYSIS**

1. Direct Stress Condition

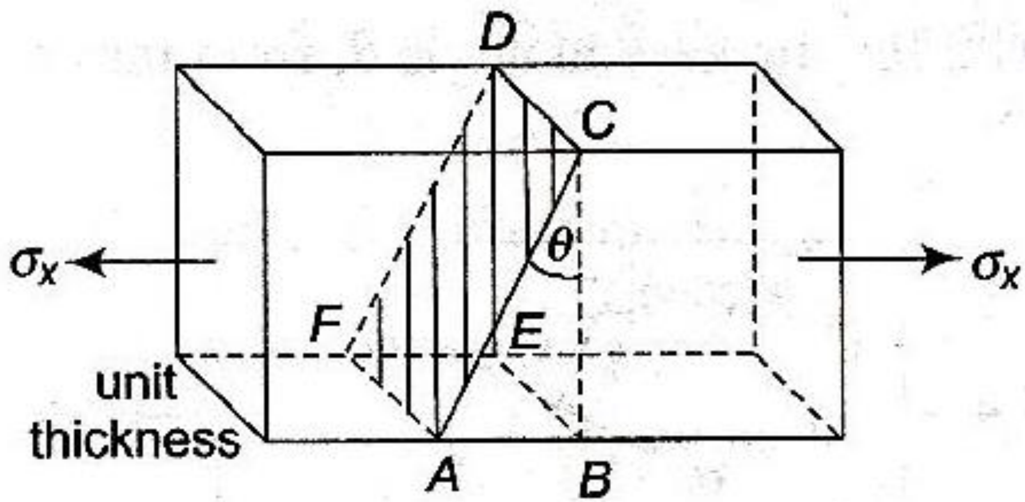
2. Bi-axial Stress Condition

3. Pure Shear Stress Condition

4. Bi-axial And Shear Stress Condition

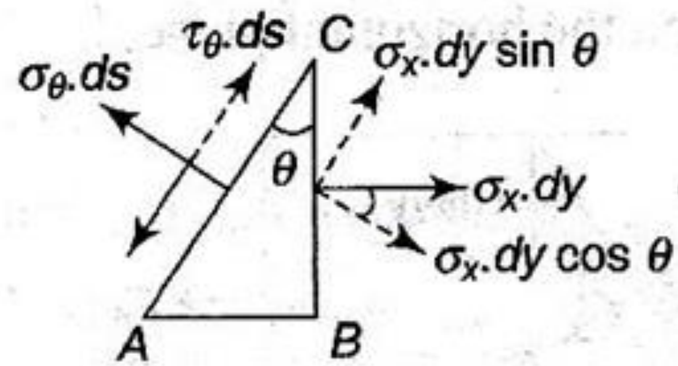
DIRECT STRESS CONDITION

(Refer VIET note)



(a)

dy = the length of the side BC
ds = the length of the side AC
σ_x = normal stress acting on the plane $BCDE$
σ_θ = normal stress acting on the plane $ACDF$
τ_θ = tangential or shear stress acting on the plane $ACDF$



$$\sigma_\theta \cdot ds - \sigma_x.dy.\cos\theta = 0$$

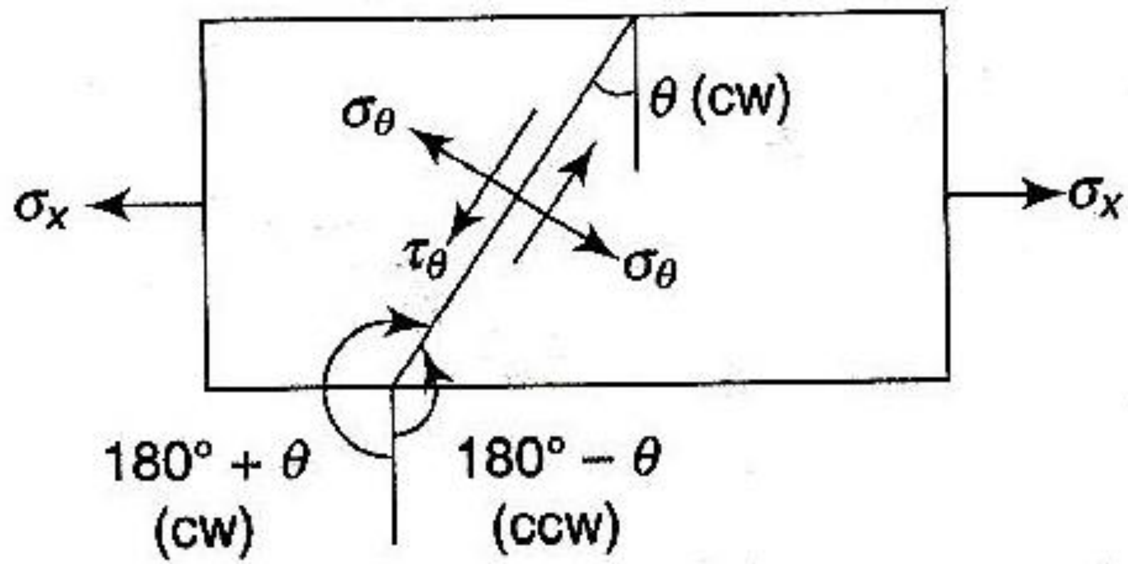
$$\sigma_\theta = \frac{\sigma_x\, dy \cos\theta}{ds} = \frac{\sigma_x\, dy \cos\theta}{dy/\cos\theta} = \sigma_x \cos^2\theta$$

$$\tau_\theta \cdot ds + \sigma_x .dy.\sin\theta = 0$$

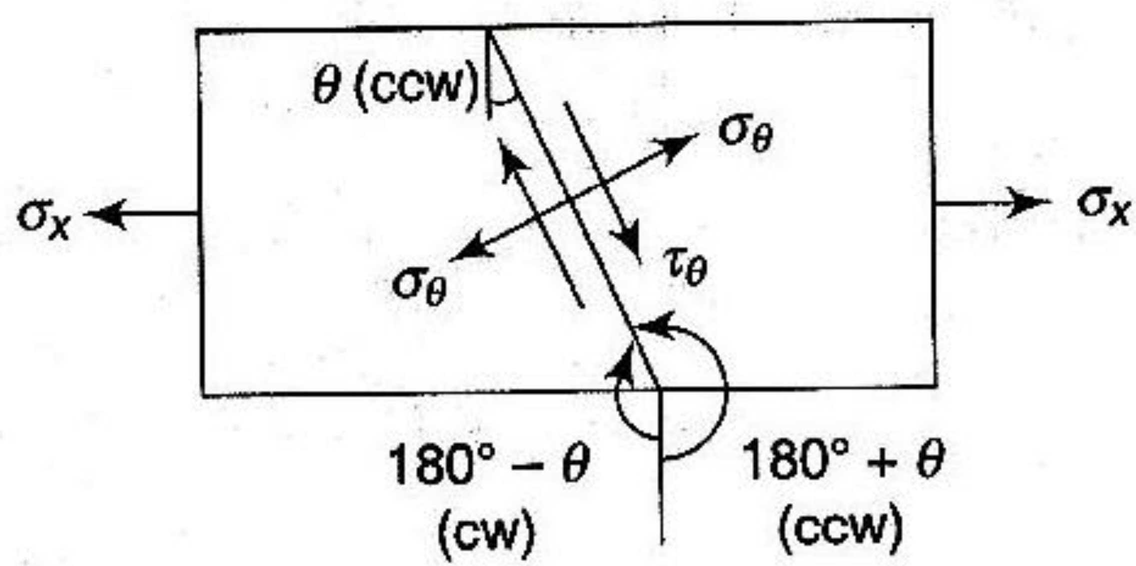
$$\tau_\theta = -\frac{\sigma_x dy \sin\theta}{ds} = -\frac{\sigma_x dy \sin\theta}{dy/\cos\theta} = -\sigma_x \sin\theta\cos\theta = -\frac{1}{2}\sigma_x \sin 2\theta$$

$\boxed{\sin(2\theta) = 2 \sin\theta \cos\theta}$

- When $\theta = 0°$, $\sigma_\theta = \sigma_x$ $\tau_\theta = 0$
- When $\theta = 45°$, $\sigma_\theta = \sigma_x/2$ $\tau_\theta = -\sigma_x/2$ (maximum, counter-clockwise)
- When $\theta = 90°$, $\sigma_\theta = 0$ $\tau_\theta = 0$
- When $\theta = 135°$ $\sigma_\theta = \sigma_x/2$ $\tau_\theta = \sigma_x/2$ (maximum, clockwise)


θ (cw)
σ_θ
σ_x
σ_x
τ_θ
σ_θ
$180° + \theta$
(cw)
$180° - \theta$
(ccw)

(a)


θ (ccw)
σ_θ
σ_x
σ_x
σ_θ
τ_θ
$180° - \theta$
(cw)
$180° + \theta$
(ccw)

(b)

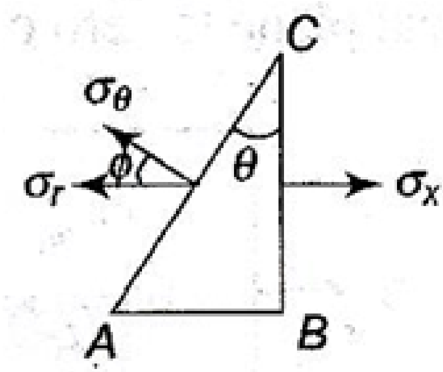
$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2}$$

$$= \sigma_x \sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}$$

$$= \sigma_x \cos\theta\sqrt{\cos^2\theta + \sin^2\theta}$$

$$= \sigma_x \cos\theta$$

$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$



$$\tan\varphi = \frac{\sigma_x \sin\theta\cos\theta}{\sigma_x \cos^2\theta} = \tan\theta$$

$$\varphi = \theta$$

EQUATIONS OF σ_θ and τ_θ

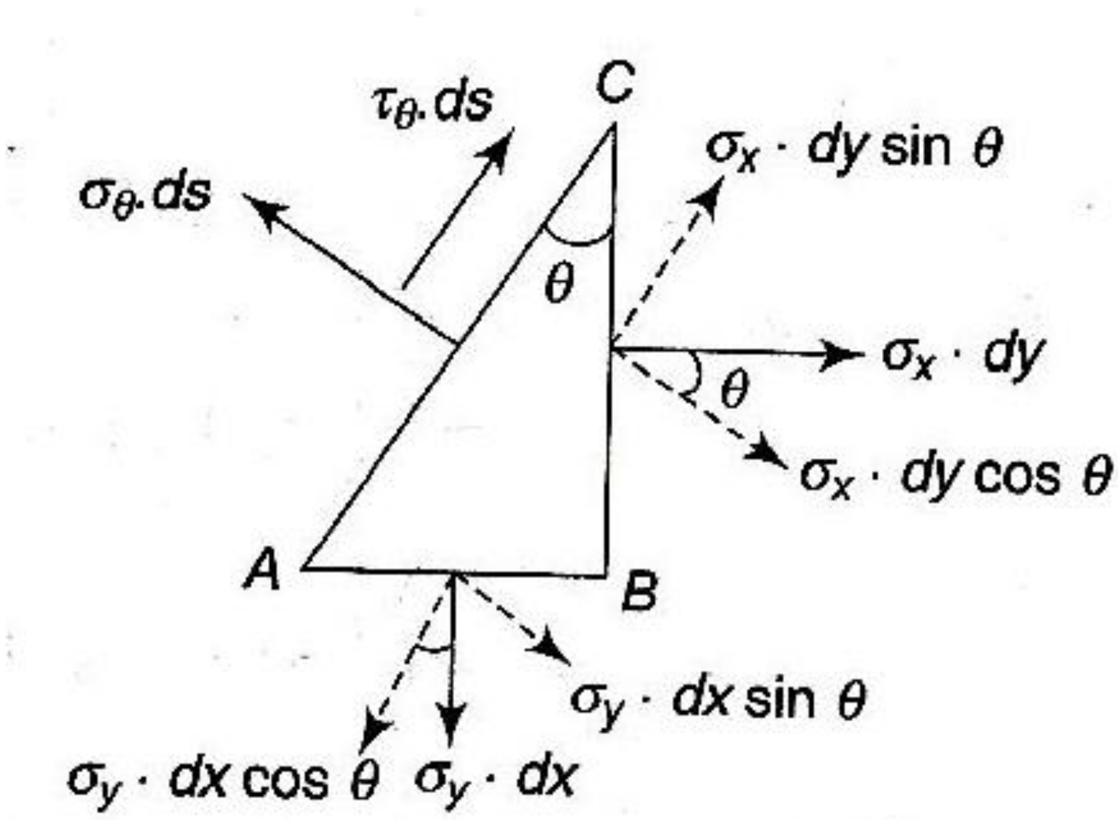
1. DIRECT STRESS CONDITION

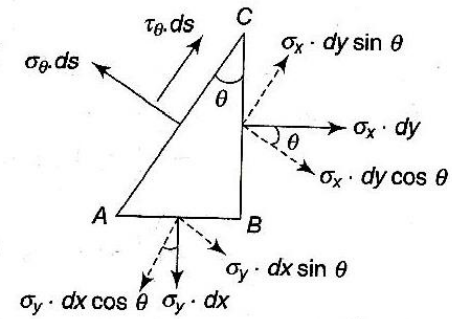
$$\sigma_\theta = \sigma_x \cos^2 \theta$$

$$\tau_\theta = -\frac{1}{2}\sigma_x \sin 2\theta$$

2. BI-AXIAL STRESS CONDITION

- Let dx, dy and ds be the lengths of sides AB, BC and AC respectively





$$\sigma_\theta \cdot ds - \sigma_x \cdot dy \cdot \cos\theta - \sigma_y \cdot dx \cdot \sin\theta = 0$$

$$\sigma_\theta = \frac{\sigma_x\, dy \cos\theta}{ds} + \frac{\sigma_y\, dx \sin\theta}{ds} = \frac{\sigma_x\, dy \cos\theta}{dy/\cos\theta} + \frac{\sigma_y\, dx \sin\theta}{dx/\sin\theta}$$

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta$$

$$\sigma_\theta = \sigma_x\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y\left(\frac{1-\cos 2\theta}{2}\right) = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta$$

Resolving the forces in the direction of τ_θ,

$$\tau_\theta \cdot ds + \sigma_x .dy \cdot \sin\theta - \sigma_y .dx \cdot \cos\theta = 0$$

$$\tau_\theta = -\frac{\sigma_x\, dy \sin\theta}{ds} + \frac{\sigma_y\, dx \cos\theta}{ds} = -\frac{\sigma_x\, dy \sin}{dy/\cos\theta} + \frac{\sigma_y\, dx \cos\theta}{dx/\sin\theta}$$

$$= -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta$$

$$= -(\sigma_x - \sigma_y)\sin\theta\cos\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

Resultant stress,

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2}$$

$$= \left[\left\{\frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta\right\}^2 + \left\{-\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta\right\}^2\right]^{1/2}$$

$$= \left[\frac{1}{4}(\sigma_x + \sigma_y)^2 + \frac{1}{4}(\sigma_x - \sigma_y)^2\cos^2 2\theta + \frac{1}{2}(\sigma_x + \sigma_y)(\sigma_x - \sigma_y)\cos 2\theta + \frac{1}{4}(\sigma_x - \sigma_y)^2\sin^2 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{4}(\sigma_x + \sigma_y)^2 + \frac{1}{4}(\sigma_x - \sigma_y)^2 + \frac{1}{2}(\sigma_x^2 - \sigma_y^2)\cos 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{4}(\sigma_x^2 + \sigma_y^2 + 2\sigma_x\sigma_y + \sigma_x^2 + \sigma_y^2 - 2\sigma_x\sigma_y) + \frac{1}{2}(\sigma_x^2 - \sigma_y^2)\cos 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{2}(\sigma_x^2 + \sigma_y^2) + \frac{1}{2}(\sigma_x^2 - \sigma_y^2)\cos 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{2}\sigma_x^2(1 + \cos 2\theta) + \frac{1}{2}\sigma_y^2(1 - \cos 2\theta)\right]^{1/2}$$

$$= \left[\frac{1}{2}\sigma_x^2 \cdot 2\cos^2\theta + \frac{1}{2}\sigma_y^2 \cdot 2\sin^2\theta\right]^{1/2}$$

$$= \sqrt{\sigma_x^2 \cos^2\theta + \sigma_y^2 \sin^2\theta}$$

the angle of inclination of the resultant with σ_θ.

$$\tan\varphi = \frac{\tau_\theta}{\sigma_\theta} = \frac{-(\sigma_x - \sigma_y)\sin\theta\cos\theta}{\sigma_x\cos^2\theta + \sigma_y\sin^2\theta} = \frac{\sigma_y - \sigma_x}{\sigma_x\cot\theta + \sigma_y\tan\theta}$$

For greatest obliquity or inclination of the resultant with the normal stress,

$$\frac{d(\tan\varphi)}{d\theta} = 0$$

$$-\sigma_x\operatorname{cosec}^2\theta + \sigma_y\sec^2\theta = 0 \quad \text{or} \quad \sigma_x\operatorname{cosec}^2\theta = \sigma_y\sec^2\theta$$

$$\tan^2\theta = \frac{\sigma_x}{\sigma_y} \quad \text{or} \quad \tan\theta = \sqrt{\frac{\sigma_x}{\sigma_y}}$$

$$\tan \varphi_{\max} = \frac{\sigma_y - \sigma_x}{\sigma_x\sqrt{\sigma_y/\sigma_x} + \sigma_y\sqrt{\sigma_x/\sigma_y}} = \frac{\sigma_y - \sigma_x}{2\sqrt{\sigma_x\sigma_y}}$$

The angle of inclination of the resultant with σ_x,

$$\tan\alpha = \frac{\sigma_y \cdot dx}{\sigma_x \cdot dy} = \frac{\sigma_y \cdot dy \cdot \tan\theta}{\sigma_x \cdot dy} = \frac{\sigma_y}{\sigma_x}\tan\theta$$

- The normal stress on the inclined plane varies between the values of σ_x and σ_y as the angle θ is increased from 0° to 90°. For equal values of the two axial stresses ($\sigma_x = \sigma_y$), σ_θ is always equal to σ_x or σ_y.

- The shear stress is zero on planes with angles 0° and 90°, i.e., on horizontal and vertical planes. It has maximum value numerically equal to one half the difference between given normal stresses which occurs on planes at ± 45° to the given planes.

$$\tau_{max} = \pm\frac{1}{2}(\sigma_x - \sigma_y)$$

and the normal stress across the same plane,

$$\sigma_{45°} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 90° = \frac{1}{2}(\sigma_x + \sigma_y)$$

- Shear stress in a body subjected to two equal perpendicular stresses is zero
- If any of the given stresses is compressive, the stress can be replaced by a negative sign in the above derived expressions, i.e., σ_x with $-\sigma_x$ and σ_y with $-\sigma_y$.

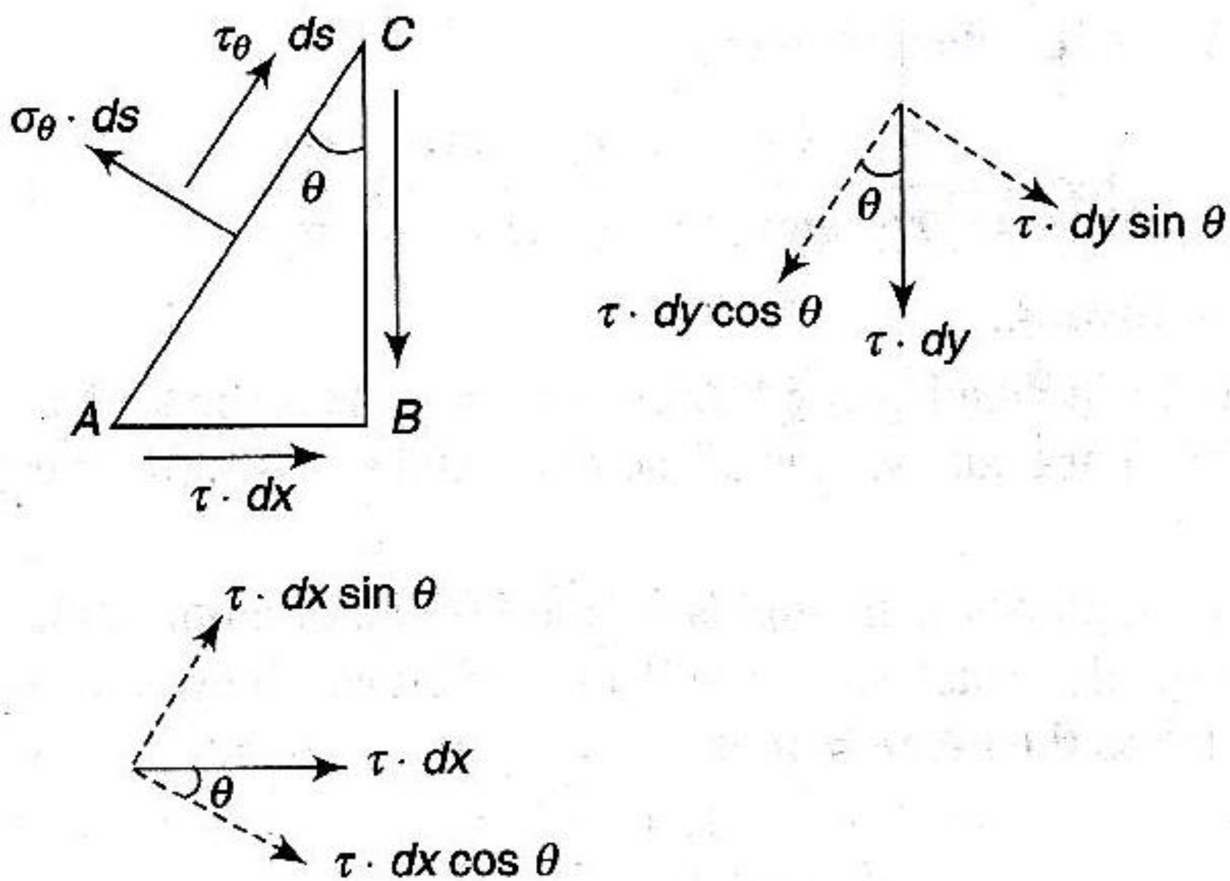
EQUATIONS OF σ_θ and τ_θ

BI-AXIAL STRESS CONDITION

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta$$

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

PURE SHEAR STRESS CONDITION



$$\sigma_\theta \cdot ds - \tau \cdot dx \cdot \cos\theta - \tau \cdot dy \cdot \sin\theta = 0$$

$$\sigma_\theta = \frac{dx \cos\theta}{ds} + \frac{\tau\, dy \sin\theta}{ds} = \frac{\tau\, dx \cos\theta}{dx/\sin\theta} + \frac{\tau\, dy \sin\theta}{dy/\cos\theta}$$

$$= \tau \sin\theta \cos\theta + \tau \sin\theta \cos\theta = \tau \cdot \sin 2\theta$$

Resolving the forces in the direction of τ_θ,

$$\tau_\theta \cdot ds - \tau \cdot dy \cdot \cos\theta + \tau \cdot dx \cdot \sin\theta = 0$$

$$\tau_\theta = \frac{\tau\, dy \cos\theta}{ds} - \frac{\tau\, dx \sin\theta}{ds} = \frac{\tau\, dy \cos\theta}{dy/\cos\theta} - \frac{\tau\, dx \sin}{dx/\sin\theta}$$

$$= \tau \cos^2\theta - \tau \sin^2\theta = \tau\left[\left(\frac{1+\cos 2\theta}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\right] = \tau \cos 2\theta$$

The resultant stress on the plane AC, $\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \tau\sqrt{(\sin 2\theta)^2 + (\cos 2\theta)^2}$

$$= \tau$$

Inclination with the direction of shear stress planes, $\tan\varphi = \dfrac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$

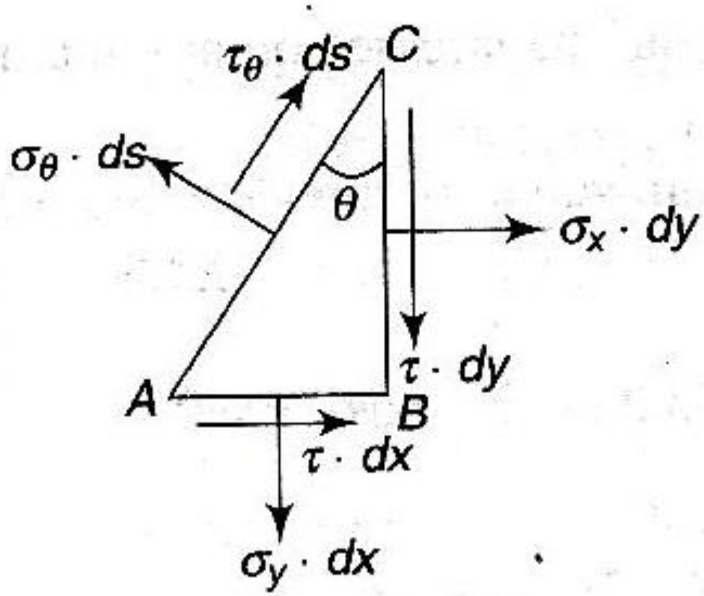
or $$\varphi = 2\theta$$

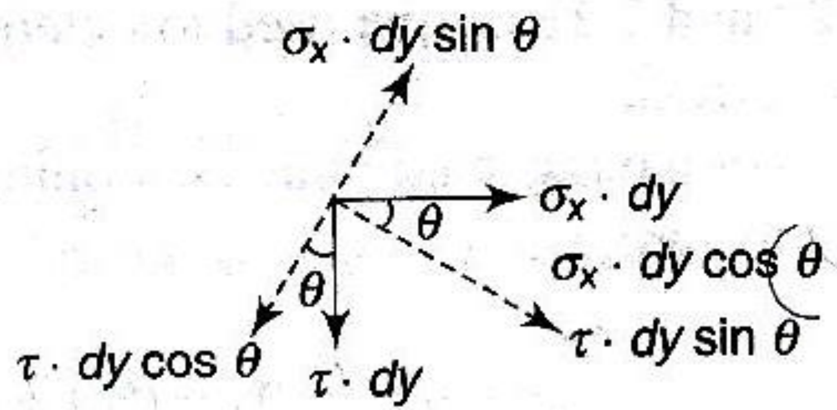
Equations for
PURE SHEAR STRESS CONDITION

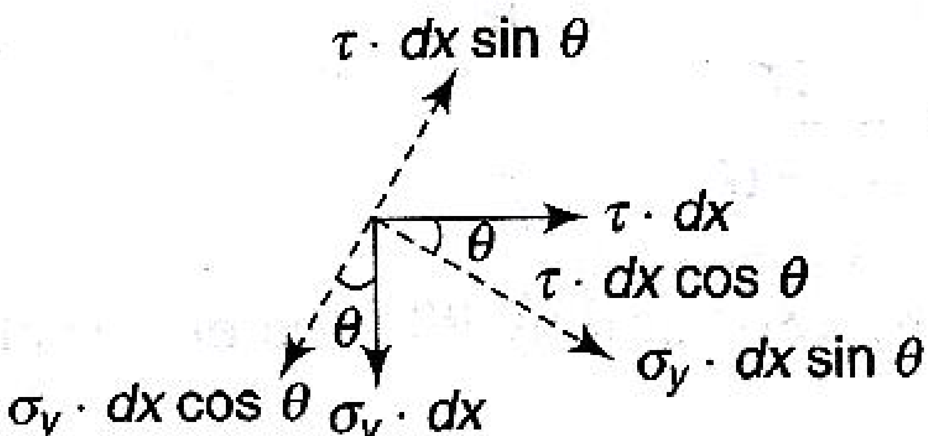
$$\sigma_\theta = \tau \cdot \sin 2\theta$$

$$\tau_\theta = \tau \cos 2\theta$$

BI-AXIAL AND SHEAR STRESS CONDITION







$$\sigma_\theta \cdot ds = \sigma_x \cdot dy \cdot \cos\theta + \sigma_y \cdot dx \cdot \sin\theta + \tau \cdot dy \cdot \sin\theta + \tau \cdot dx \cdot \cos\theta$$

$$\sigma_\theta = \frac{\sigma_x\, dy \cos\theta}{ds} + \frac{\sigma_y\, dx \sin\theta}{ds} + \frac{\tau \cdot dy \sin\theta}{ds} + \frac{\tau \cdot dx \cos\theta}{ds}$$

$$= \frac{\sigma_x\, dy \cos\theta}{dy/\cos\theta} + \frac{\sigma_y\, dx \sin\theta}{dx/\sin\theta} + \frac{\tau\, dy \sin\theta}{dy/\cos\theta} + \frac{\tau\, dx \cos\theta}{dx/\sin\theta}$$

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin\theta \cos\theta + \tau \sin\theta \cos\theta$$

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin 2\theta$$

$$= \sigma_x \left(\frac{1+\cos 2\theta}{2} \right) + \sigma_y \left(\frac{1-\cos 2\theta}{2} \right) + \tau \cdot \sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \cdot \sin 2\theta$$

Resolving the forces in the direction of τ,

$$\tau\theta \cdot ds + \sigma_x \cdot dy \cdot \sin\theta - \sigma_y \cdot dx \cdot \cos\theta - \tau \cdot dy \cdot \cos\theta + \tau \cdot dx \cdot \sin\theta = 0$$

$$\tau_\theta = -\frac{\sigma_x\, dy \sin\theta}{ds} + \frac{\sigma_y\, dx \cos\theta}{ds} + \frac{\tau \cdot dy \cos\theta}{ds} - \frac{\tau \cdot dx \sin\theta}{ds}$$

$$= -\frac{\sigma_x\, dy \sin}{dy/\cos\theta} + \frac{\sigma_y\, dx \cos\theta}{dx/\sin\theta} + \frac{\tau\, dy \cos\theta}{dy/\cos\theta} - \frac{\tau\, dx \sin}{dx/\sin\theta}$$

$$= -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau\cos^2\theta - \tau\sin^2\theta$$

$$= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\left[\left(\frac{1+\cos 2\theta}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\right]$$

$$= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta$$

To determine the planes having maximum and minimum values of direct stress,

$$\frac{d\sigma_\theta}{d\theta} = 0 - \frac{1}{2}(\sigma_x - \sigma_y)\,2\sin 2\theta + 2\tau\cdot\cos 2\theta = 0$$

$$\frac{1}{2}(\sigma_x - \sigma_y)2\sin 2\theta = 2\tau\cdot\cos 2\theta$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y}$$

Equations for
BI-AXIAL AND SHEAR STRESS CONDITION

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \cdot \sin 2\theta$$

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau \cos 2\theta$$

SUM OF DIRECT STRESSES ON TWO MUTUALLY PERPENDICULAR PLANES

Direct stress on an inclined plane at angle θ is given by,

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \cdot \sin 2\theta$$

Direct stress on an inclined plane at angle $(\theta+90°)$ will be,

$$\sigma_{(\theta+90°)} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2(\theta + 90°) + \tau \cdot \sin 2\,(\theta + 90°)$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau \cdot \sin 2\theta$$

$$\sigma_\theta + \sigma_{(\theta+90°)} = \sigma_x + \sigma_y$$

PRINCIPAL PLANES
AND
PRINCIPAL STRESSES

PRINCIPAL PLANES

Planes which have no shear stress. These planes

carry only normal stresses

PRINCIPAL STRESSES

Normal stresses acting on principal planes

- ✓ Larger of two stresses

σ_1 – Major principal stress

- ✓ Smaller

σ_2 – Minor Principal stress

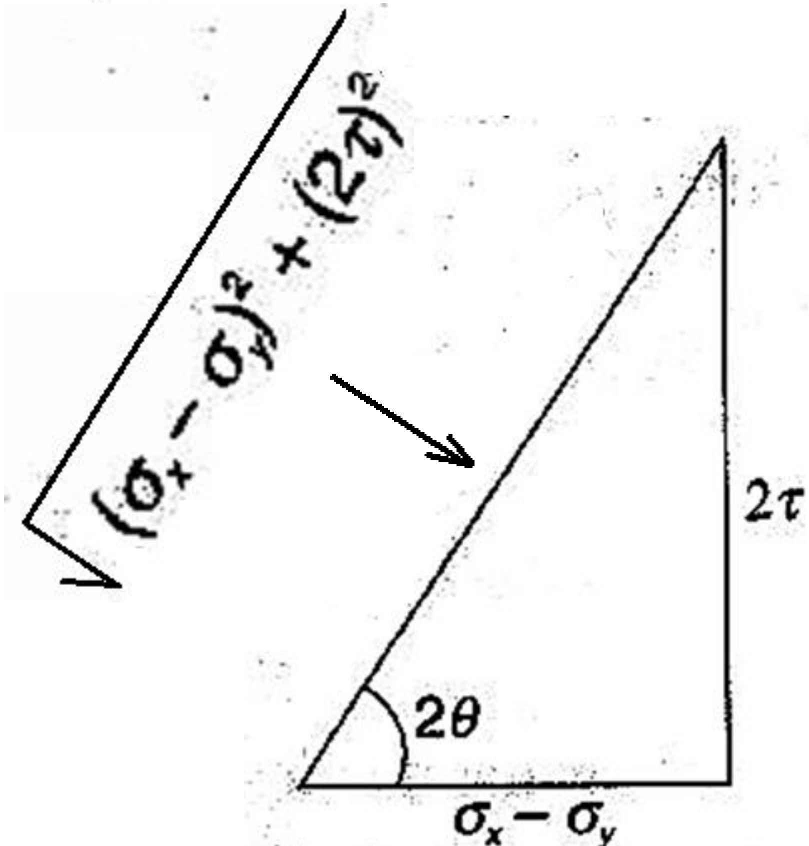
- ✓ Corresponding planes are **MAJOR and MINOR principal planes**

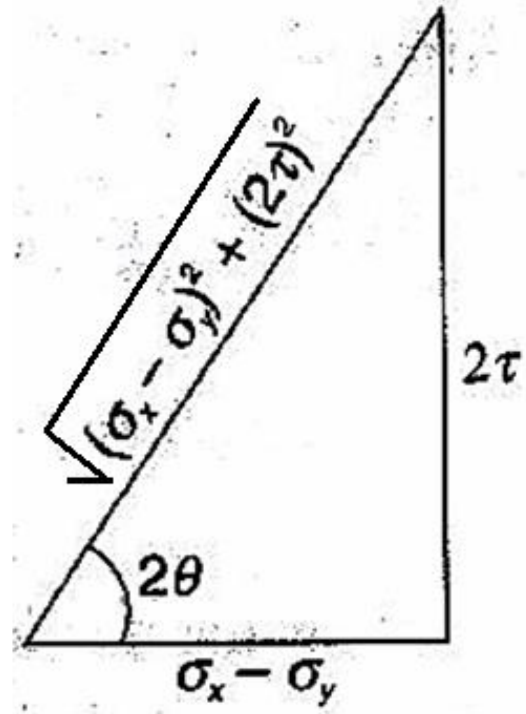
As shear stress is zero in principal planes,

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta = 0$$

$$\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta = \tau\cos 2\theta$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y}$$





$$\sin 2\theta = \pm \frac{2\tau}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$\cos 2\theta = \pm \frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \cdot \sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\frac{(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}} \pm \tau \,.\, \frac{2\tau}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

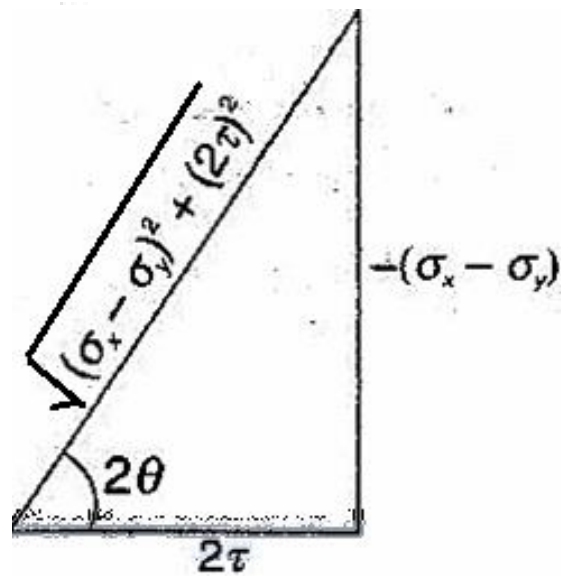
MAXIMUM (PRINCIPAL) SHEAR STRESSES

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau \cos 2\theta$$

For maximum value of τ_θ, differentiate it with respect to θ and equate to zero,

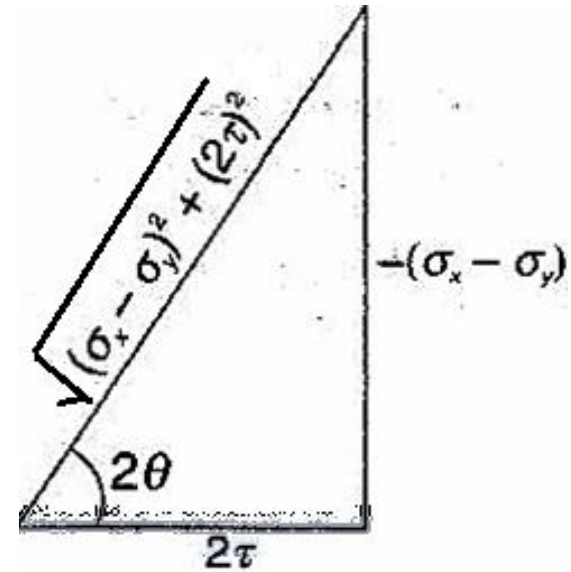
$$\frac{d\tau_\theta}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau \sin 2\theta = 0$$

$$\tan 2\theta = -\frac{\sigma_x - \sigma_y}{2\tau}$$



$$\sin 2\theta = \mp \frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$\cos 2\theta = \pm \frac{2\tau}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}}$$



$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta$$

$$= -\left(\mp\frac{1}{2}(\sigma_x - \sigma_y)\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}\right) \pm \tau \cdot \frac{2\tau}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \pm\frac{1}{2}\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

$$\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

maximum principal stress, $\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

minimum principal stress, $\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

Subtracting (ii) from (i), $\sigma_1 - \sigma_2 = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2)$$

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

principal planes are given by, $\tan 2\theta_p = \dfrac{2\tau}{\sigma_x - \sigma_y}$

planes of maximum shear stress, $\tan 2\theta_s = -\dfrac{\sigma_x - \sigma_y}{2\tau}$

$\tan 2\theta_p \cdot \tan 2\theta_s = -1$ which means $2\theta_s = 2\theta_p + 90°$ or $\theta_s = \theta_p + 45°$

indicates that the planes of maximum shear stress lie at 45° to the planes of principal stresses

NORMAL STRESS ON THE PLANES OF MAXIMUM SHEAR STRESS

Direct stress on an inclined plane is given by

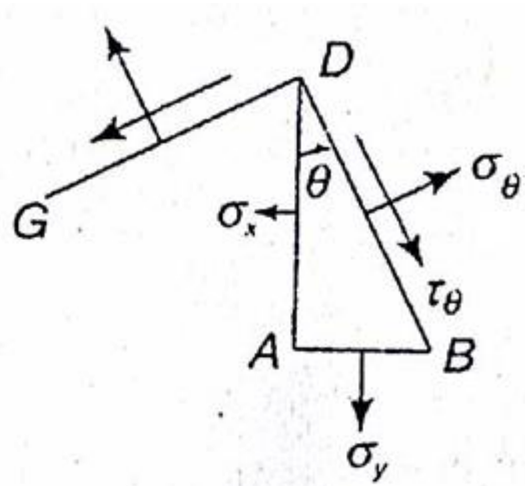
$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \cdot \sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \cos 2\theta\left[\frac{1}{2}(\sigma_x - \sigma_y) + \tau \cdot \tan 2\theta\right]$$

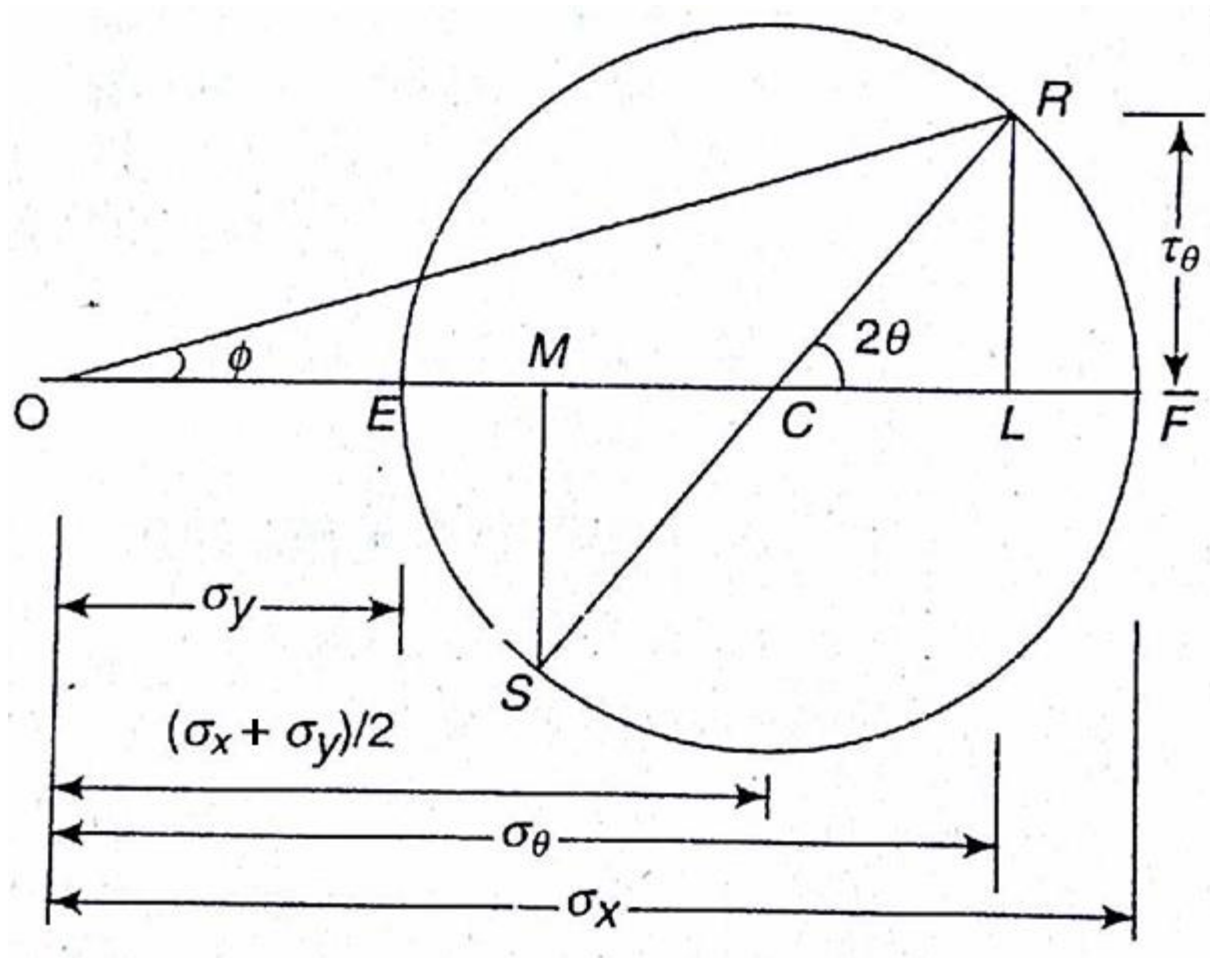
$$= \frac{1}{2}(\sigma_x + \sigma_y) + \cos 2\theta\left[\frac{1}{2}(\sigma_x - \sigma_y) - \tau \cdot \frac{\sigma_x - \sigma_y}{2\tau}\right]$$

$$= \frac{1}{2}(\sigma_x + \sigma_y)$$

Mohr’s stress circle



D
G
A
B
θ
σ_θ
σ_x
τ_θ
σ_y



R
M
2θ
φ
O
E
C
L
F
τ_θ
S
σ_y
$(\sigma_x + \sigma_y)/2$
σ_θ
σ_x

$$OC = \frac{1}{2}(OC + OC) = \frac{1}{2}(OF - CF) + (OE + CE)$$

$$= \frac{1}{2}(OF - CF) + (OE + CF) \qquad \ldots\ldots(CE = CF)$$

$$= \frac{1}{2}(OF + OE) = \frac{1}{2}(\sigma_x + \sigma_y)$$

$$CL = CR \cos 2\theta = CF \cos 2\theta = \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta \qquad (CR = CF)$$

Thus $OL = OC + CL = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta = \sigma_\theta$

And $LR = CR \sin 2\theta = CF \sin 2\theta = \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta = \tau_\theta$

A positive expression of τ_θ indicates it is clockwise for counter-clockwise angle θ.

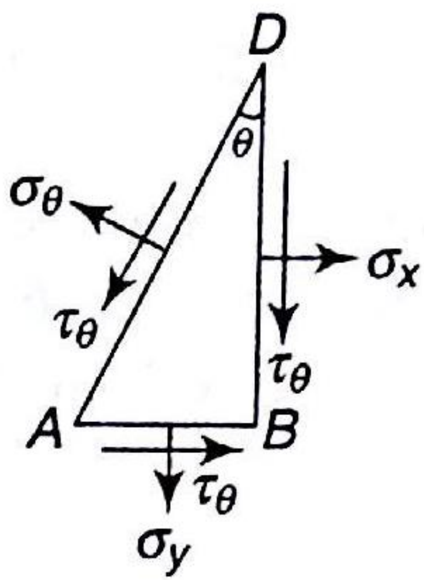
The resultant of OL and LR is represented by OR at an angle φ with the OL, i.e., with the direction of σ_θ.

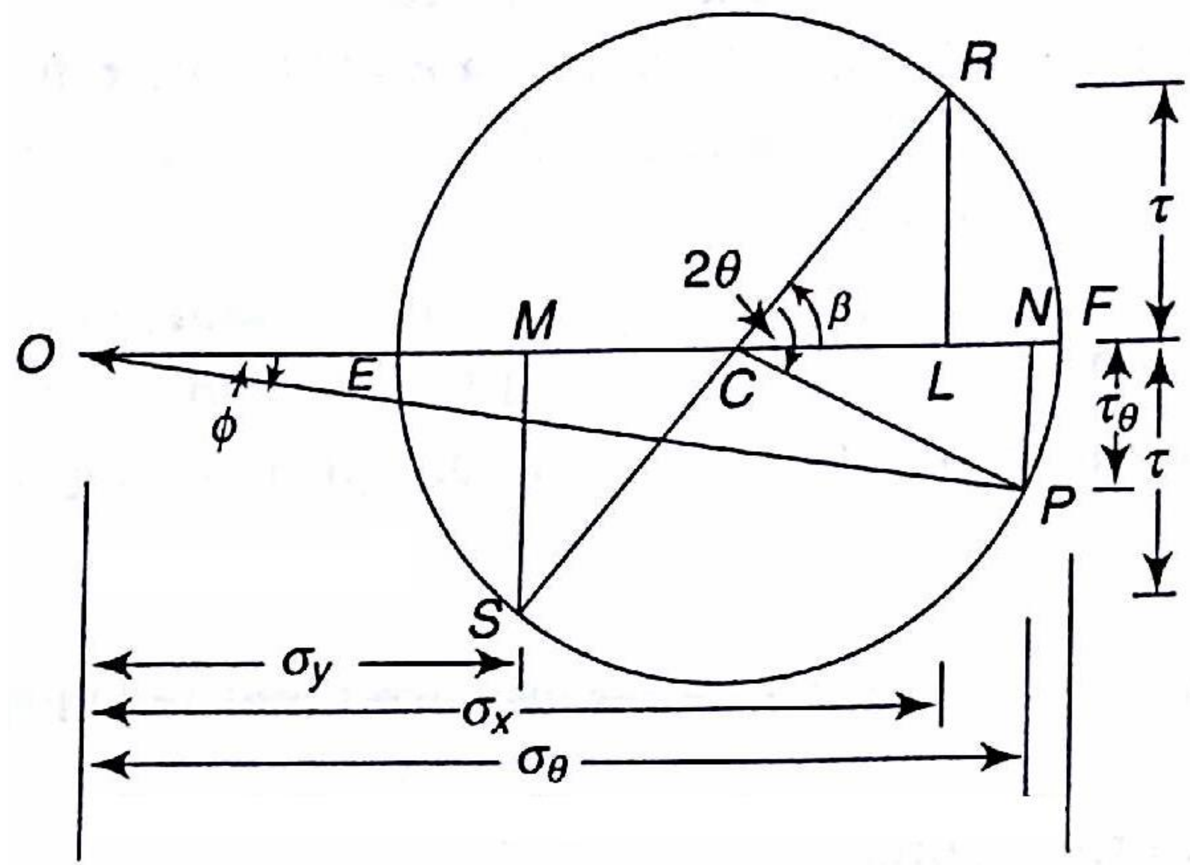
- Direct stress component on the inclined plane BD represented by OR is on the right side of the origin, it is positive or tensile.

- Shear stresses giving a clockwise rotation are assumed positive and are above the x-axis. In the present case, the shear component LR represents a clockwise direction.

- The stress components on a plane DG perpendicular to BD are obtained by rotating the radial line CR through double the angle, i.e., 180° in clockwise or counter-clockwise direction. Thus, CS represents the plane DG. OM indicates the tensile component and the SM the shear component.

Two Direct stresses with simple shear



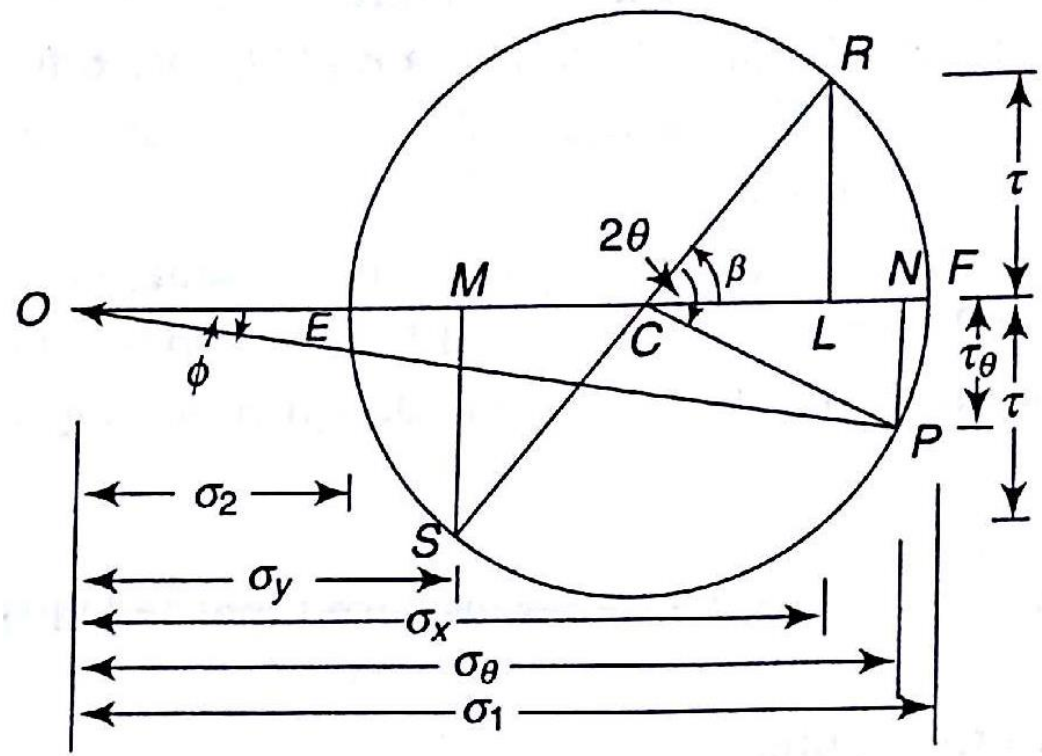


Principal Stresses

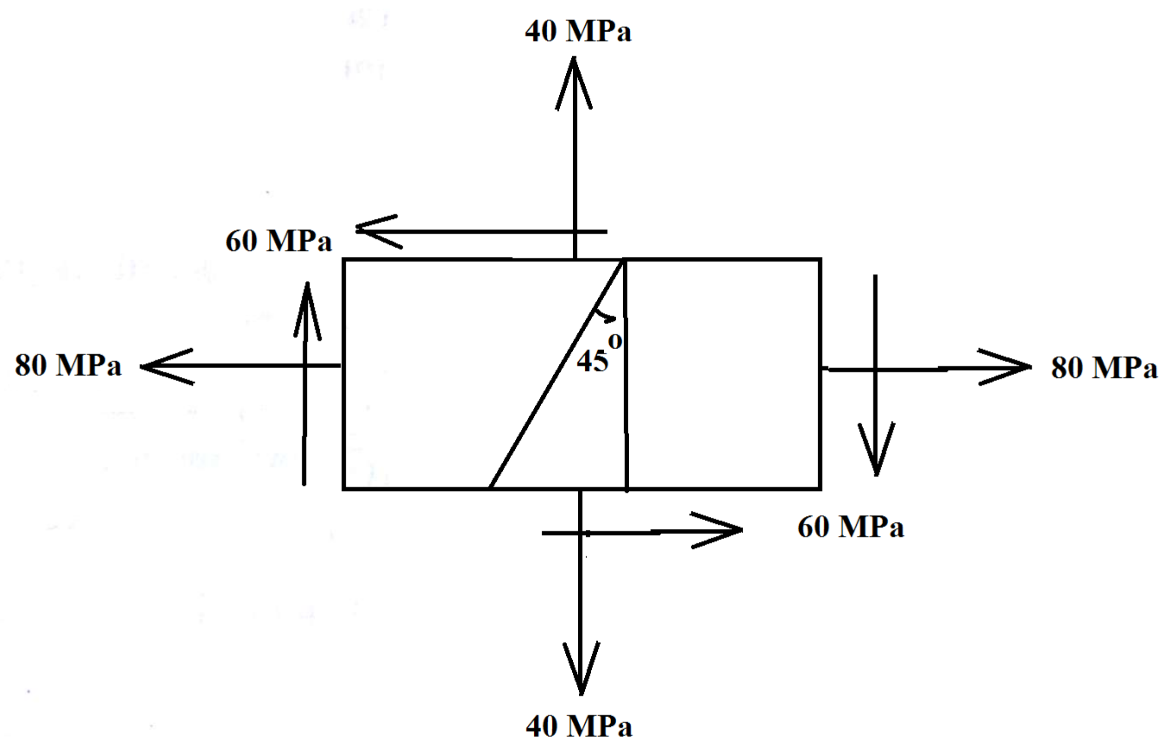
As shear stress is zero on the principal planes, *OF* represents the major principal plane with maximum normal stress.

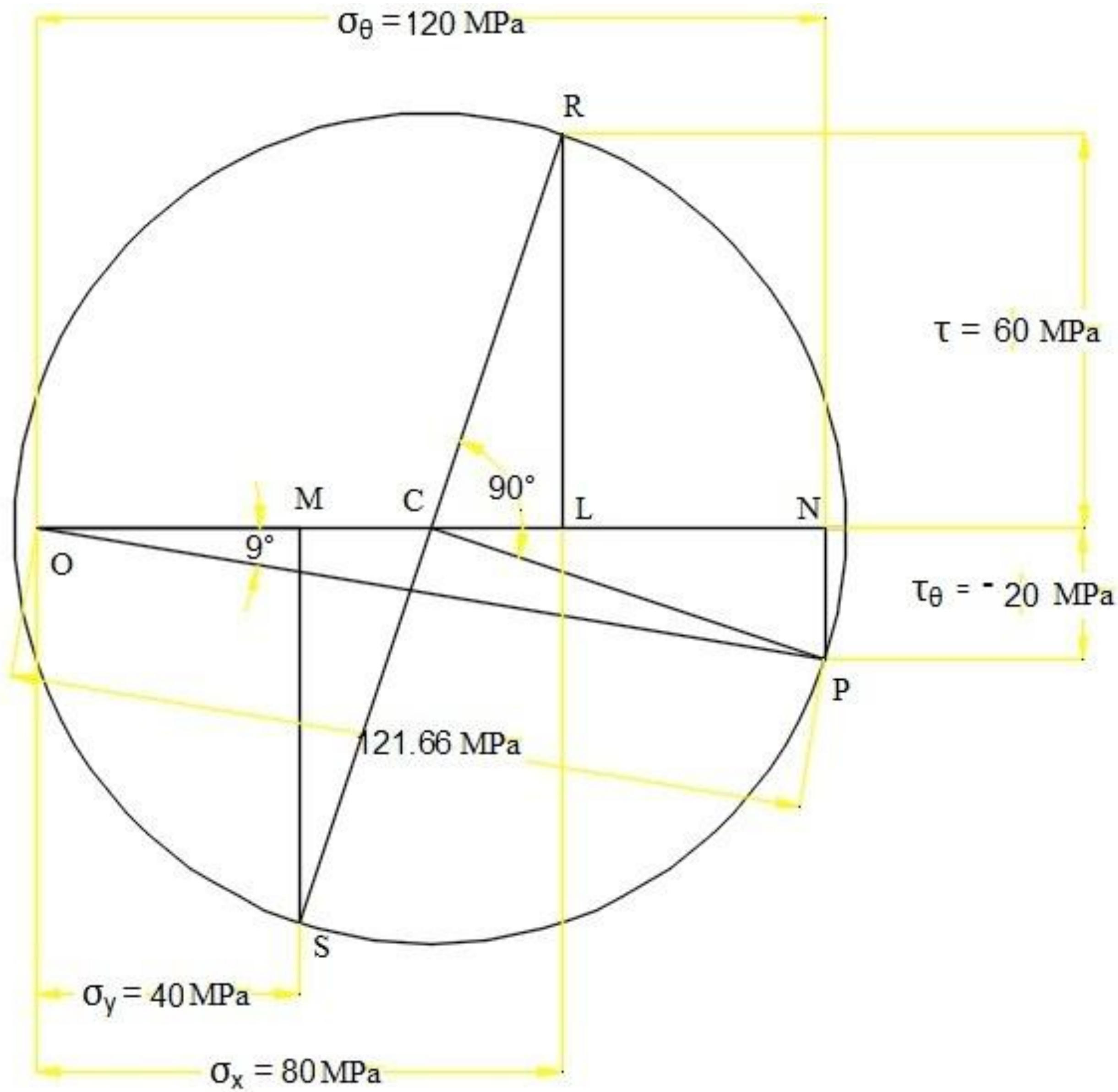
In a similar way, *OE* represents the minor principal plane.

The angles of inclination of planes of major and minor principal stresses are $\beta/2$ and $(90° + \beta/2)$ respectively clockwise with the plane of stress σ_x.



Q). For the stress system shown in the diagram, find the normal and shear stresses acting on the inclined plane using Mohr's circle method. Also find the resultant stress and its direction

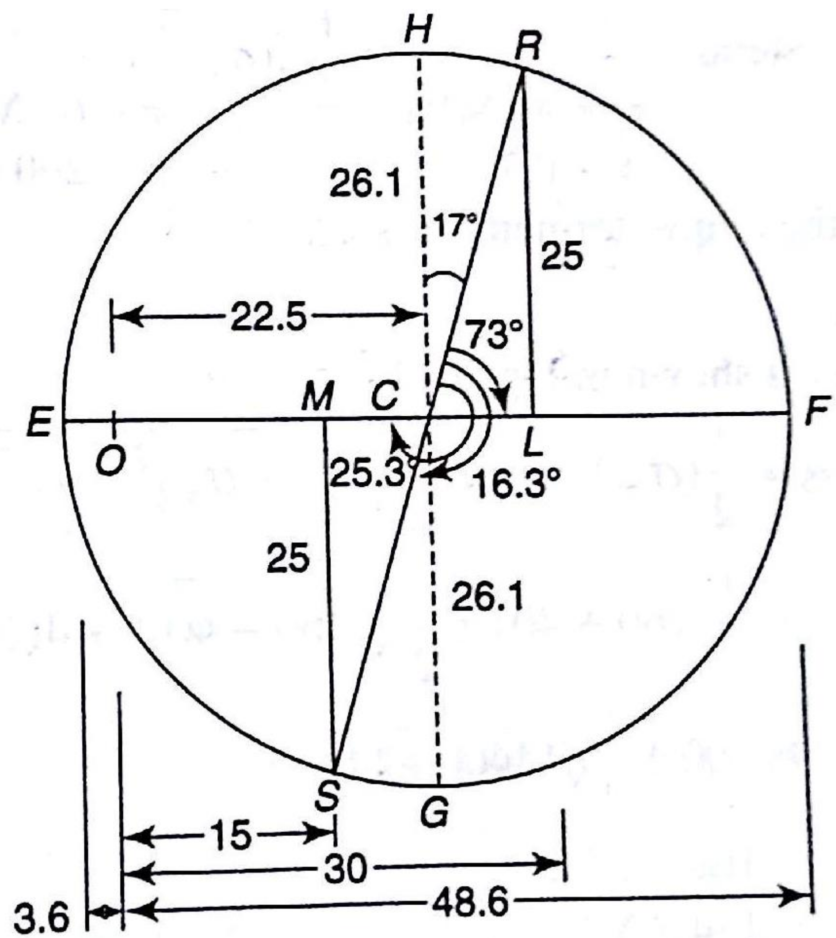




σθ = 120 MPa
R
τ = 60 MPa
90°
M
C
L
N
9°
O
τθ = - 20 MPa
P
121.66 MPa
S
σy = 40 MPa
σx = 80 MPa

The stresses on two perpendicular planes through a point in a body are 30 MPa and 15 MPa both tensile along with a shear stress of 25 MPa. Find

(i) the magnitude and direction of principal stresses
(ii) the planes of maximum shear stress
(iii) the normal and shear stresses on the planes of maximum shearing stress



Plane state of strain

- In a plane strain problem $\varepsilon_{zz} = 0$, $\gamma_{xz} = \gamma_{yz} = 0$ and only 3 components of strain ε_{xx} , ε_{yy} and γ_{xy} are enough to describe the state of strain.

- If direct and shear strains along x and y directions are known, normal strain (ε_θ) and the shear strain (ϕ_θ) at angle θ with the x direction of a body can be found.

- Normal Strain

$$\varepsilon_\theta = \varepsilon_x . \cos^2\theta + \varepsilon_y . \sin^2\theta + \varphi . \sin\theta . \cos\theta$$

$$\varepsilon_\theta = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{1}{2}\varphi \sin 2\theta$$

- Analogous to normal stress equation

- In a linear strain system, $\varepsilon_\theta = \varepsilon_x \cdot \cos^2\theta$ or $\varepsilon_x\left(\frac{1+\cos 2\theta}{2}\right)$
- In a pure shear system and for $\theta = 45°, \varepsilon_{45°} = \varphi/2$.

- Shear strain $\varphi_\theta = (\varepsilon_x - \varepsilon_y)\sin 2\theta - \varphi \cos 2\theta$

Principal Strains

- The maximum and minimum values of strains on any plane at a point are known as the principal strains
- Corresponding planes – Principal planes for strains

$$\text{Principal strain} = \frac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \varphi^2}$$

- Principal planes are given by $\tan 2\theta = \dfrac{\varphi}{\varepsilon_x - \varepsilon_y}$

- Principal (Maximum) shear strains

$$\varphi_{\max} = \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma^2}$$

- Planes of maximum shear strains

$$\tan 2\theta = -\frac{\varepsilon_x - \varepsilon_y}{\varphi}$$

- Sum of direct strains on 2 mutually perpendicular planes

$$\varepsilon_\theta + \varepsilon_{\theta+90^\circ} = \varepsilon_x + \varepsilon_y$$

Principal stresses from Principal strains

In a two-dimensional system $\sigma_3 = 0$, and

$$\varepsilon_1 = \sigma_1/E - \nu\,\sigma_2/E$$

$$\varepsilon_1 \cdot E = \sigma_1 - \nu\,\sigma_2$$

$$\sigma_1 = \varepsilon_1 \cdot E + \nu\,\sigma_2 \quad \text{———} \quad \text{(i)}$$

and $\quad \varepsilon_2 = \sigma_2/E - \nu\,\sigma_1/E$

$$\varepsilon_2 \cdot E = \sigma_2 - \nu\,\sigma_1 \quad \text{———} \quad \text{(ii)}$$

Inserting the value of σ_1

$$\varepsilon_2 \cdot E = \sigma_2 - \nu\,(\varepsilon_1 \cdot E + \nu\,\sigma_2) = \sigma_2(1 - \nu^2) - \nu\,\varepsilon_1 \cdot E$$

$$\sigma_2 = \frac{E(\nu\varepsilon_1 + \varepsilon_2)}{1 - \nu^2}$$

Similarly, $\quad \sigma_1 = \dfrac{E(\nu\varepsilon_2 + \varepsilon_1)}{1 - \nu^2}$

- 3D system

$$\sigma_1 = \frac{E[(1-\nu)\varepsilon_1 + (\varepsilon_2 + \varepsilon_3)\nu]}{(1+\nu)(1-2\nu)}$$

$$\sigma_2 = \frac{E[(1-\nu)\varepsilon_2 + (\varepsilon_3 + \varepsilon_1)\nu]}{(1+\nu)(1-2\nu)}$$

$$\sigma_3 = \frac{E[(1-\nu)\varepsilon_3 + (\varepsilon_1 + \varepsilon_2)\nu]}{(1+\nu)(1-2\nu)}$$

Strain Rosette

- Strain in any direction can be measured by using an instrument known as strain gauge.
- In case, directions of principal strains are known, two strain gauges can be used to measure the strains in these directions and by using above equations, the principal stresses can be calculated.
- However, many times the directions of the principal strains are not known.
- In such cases, a set of three strain gauges, known as a **strain rosette**, can be used to find the strain in three known directions in order to determine stress condition at a point under consideration.
- Let ε_x and ε_y be the linear strains in x- and y-directions and ϕ be the shear strain at the point under consideration.
- Then linear strains in any three arbitrary chosen directions at angles θ_1, θ_2 and θ_3 made with the x-axis will be

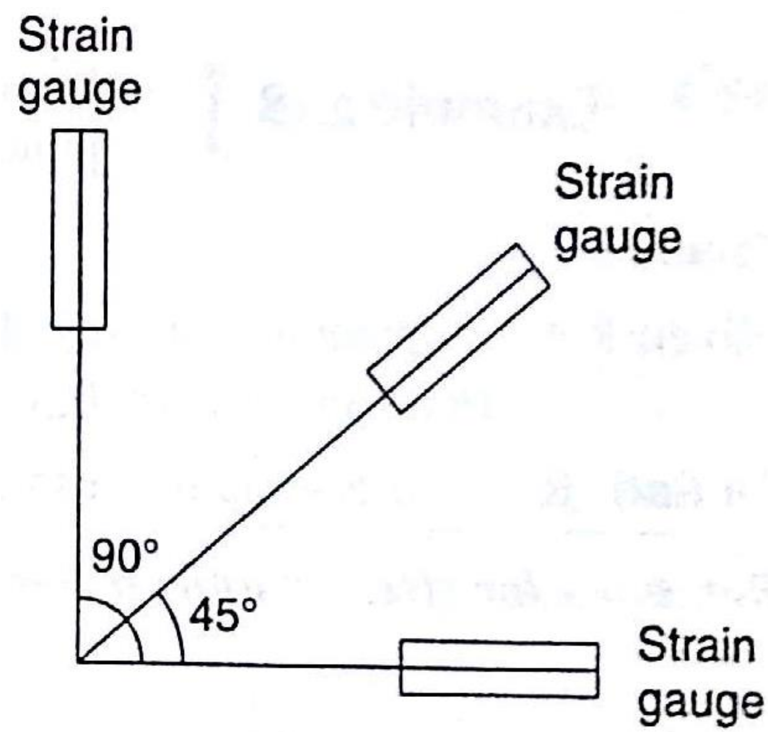
$$\varepsilon_\theta = \varepsilon_x . \cos^2\theta + \varepsilon_y . \sin^2\theta + \varphi . \sin\theta . \cos\theta$$

$$\varepsilon_{\theta 1} = \varepsilon_x \cdot \cos^2\theta_1 + \varepsilon_y \cdot \sin^2\theta_1 + \varphi \cdot \sin\theta_1 \cdot \cos\theta_1$$
$$\varepsilon_{\theta 2} = \varepsilon_x \cdot \cos^2\theta_2 + \varepsilon_y \cdot \sin^2\theta_2 + \varphi \cdot \sin\theta_2 \cdot \cos\theta_2$$
$$\varepsilon_{\theta 3} = \varepsilon_x \cdot \cos^2\theta_3 + \varepsilon_y \cdot \sin^2\theta_3 + \varphi \cdot \sin\theta_3 \cdot \cos\theta_3$$

- If three arbitrary directions are chosen in a set manner and $\varepsilon_{\theta 1}$, $\varepsilon_{\theta 2}$ and $\varepsilon_{\theta 3}$ are measured along these directions, then ε_x , ε_y and ϕ, can be calculated by using the above equations.
- Principal strains and principal stresses can then be calculated.

Rectangular strain Rosette

- If the 3 strain gauges are set at 0˚, 45˚ and 90˚ with the X direction, it is known as rectangular or 45˚ strain rosette.



Thus in this case, $\theta_1 = 0°$, $\theta_2 = 45°$ and $\theta_3 = 90°$

The above equations can be written as

$$\varepsilon_{0°} = \varepsilon_x$$

$$\varepsilon_{45°} = \frac{1}{2}(\varepsilon_x + \varepsilon_y + \varphi)$$

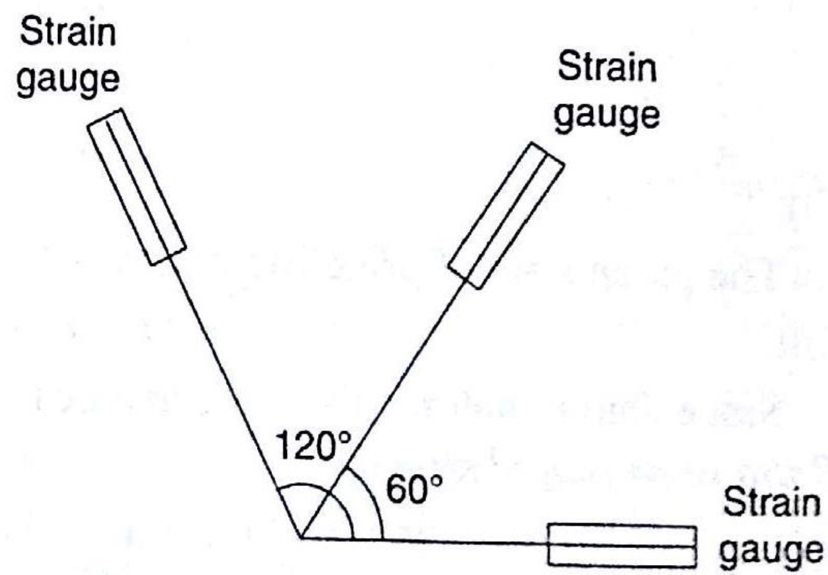
$$\varepsilon_{90°} = \varepsilon_y$$

or $\varepsilon_x = \varepsilon_{0°}$, $\varepsilon_y = \varepsilon_{90°}$ and $\varphi = 2\varepsilon_{45°} - \varepsilon_x + \varepsilon_y$

- Now principal strains and hence principal stresses can be calculated

Equiangular strain Rosette

- If the 3 strain gauges are set at 0˚, 60˚ and 120˚ with the X direction, it is known as equiangular or delta or 60˚ strain rosette.



Thus in this case, $\theta_1 = 0°$, $\theta_2 = 60°$ and $\theta_3 = 120°$

The above equations can be written as

$$\varepsilon_{0°} = \varepsilon_x \qquad \text{——— (i)}$$

$$\varepsilon_{60°} = \frac{1}{4}(\varepsilon_x + 3\varepsilon_y + \sqrt{3}\varphi) \qquad \text{——— (ii)}$$

$$\varepsilon_{120°} = \frac{1}{4}(\varepsilon_x + 3\varepsilon_y - \sqrt{3}\varphi) \qquad \text{——— (iii)}$$

From which, $\varepsilon_x = \varepsilon_{0°}$

Subtracting (iii) from (ii), $\varphi = \frac{2}{\sqrt{3}}(\varepsilon_{60°} - \varepsilon_{120°})$

From (ii), $3\varepsilon_y = 4\varepsilon_{60°} - \varepsilon_x - \sqrt{3}\varphi = 4\varepsilon_{60°} - \varepsilon_{0°} - \sqrt{3}\cdot\frac{2}{\sqrt{3}}(\varepsilon_{60°} - \varepsilon_{120°})$

or $\varepsilon_y = \frac{1}{3}(2\varepsilon_{60°} + 2\varepsilon_{120°} - \varepsilon_{0°})$

- Now principal strains and hence principal stresses can be calculated